Remarks on General Principally Injective Rings

Hasan Ögünmez

Afyon Kocatepe University, Faculty of Arts and Sciences, Department of Mathematics, Ahmet Necdet Sezer Campus, Afyonkarahisar, Turkey

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Yazısalardan Sorumlu Yazar (Corresponding author): hogueunmez@aku.edu.tr
+90 272 228 13 12 / 256 +90 272 228 12 35

ABSTRACT
In [3], Chen and Li proved that every left CS and left p-injective ring is a QF-ring. In this study, we show that a right Noetherian, left CS and left GP-injective ring is right Artinian. We also prove that, if every singular simple right \( R \)-module is GP-injective, then \( J(R) \cap Z_r = 0 \). This gives a partially answer to a question of Ming [5].

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ÖZET

Anahtar Kelimeler: CS-halkalar, GP-injektif halkalar, Noetherian, Artinian

1. INTRODUCTION
Throughout this paper, we assume that \( R \) is an associative ring (not necessarily commutative) with unity and \( M_R \) (resp., \( R/M \)) is unital right (resp. left) \( R \)-module. The notions, “\( \leq \)” will denote a submodule “\( \leq e \)” an essential submodule and \( l_R(X) \) (resp., \( r_R(X) \)) the left (resp. right) annihilator of a subset \( X \) of \( R \), respectively. We also write “\( J \)”, “\( Z_R \)” (“\( Z_l \)”), and \( S_R \) (“\( S_r \)”) for the Jacobson radical, the right (left) singular ideal and the right (left) socle of \( R \), respectively. The texts by Anderson and Fuller [1] and [6] are the general
A module $M$ is called principally injective ($p$-injective for short) if every $R$-homomorphism from a principal right ideal $aR$ to $M$ extends to one from $R$ to $M$, i.e., is given by left multiplication by an element of $M$. This is equivalent to saying that $l_{M}r_{R}(a) = Ma$ for all $a \in R$. $R$ is called right $P$-injective ring, if $R_{R}$ is a $p$-injective module. A ring $R$ is said to be general right principally injective (briefly right GP-injective) if, for any $0 \neq a \in R$, there exists a positive integer $n = n(a)$ such that $a^{n} \neq 0$ and any right $R$-homomorphism from $a^{n}R$ to $R$ extends to an endomorphism of $R$ (see [10]).

A module $M$ is called extending (or CS) if, for all $N \leq M$, there exists a direct summand $N' \leq_{d} M$ such that $N \leq c_{e} N'$ and a ring $R$ is called right (resp., left) CS if $R_{R}$ (resp., $_{R}R$) is CS (see [6]). Examples of extending modules are injective modules, quasi-injective modules and uniform modules. The notions of $p$-injective rings, CS rings and GP-injective rings have been the focus of a number of research papers.

A right $R$-module $M_{R}$ is called mininjective if, for each simple right ideal $K$ of $R$, every $R$-morphism $\alpha : K \rightarrow M$ extends to $R$; equivalently if $\alpha : m$ is left multiplication by some element $m$ of $M$. Hence the ring $R$ is right mininjective if $R_{R}$ is mininjective [8]. By [8, Lemma 1.1], $R$ is right mininjective if and only if, for $a \in R$, $l_{R}(a) = Ra$ where $Ra$ is simple right ideal of $R$. A ring $R$ is called right simple injective if for some $R$-homomorphism $\gamma$ with $\gamma(I)$ simple extends to $R$. So we have the following strict hierarchy.

\{right self injective\} $\subset$ \{right simple injective\} $\subset$ \{right mininjective\}

A ring $R$ is called a right generalized V-ring if every singular simple right $R$-module is injective.

In this paper, by using a method due to Chen and Li [3], we obtain that if $R$ is a right Noetherian, left CS and left GP-injective ring, then $R$ is right Artinian. We also prove that a right CF, right GP-injective and semi regular ring is a QF-ring.

2. RESULTS

Lemma 2.1. Let $R$ be a right Noetherian, left GP-injective and left finite dimensional ring. Then $R$ is right Artinian.

Proof. By [2, Theorem 4.6], every left GP-injective and left finite dimensional ring is semilocal. Note that in [2, Theorem 4.6], the reader is referred to [7, Theorem 3.3]. Now, because $R$ is right Noetherian, there exists $n \geq 1$ such that $l(J^n) = l(J^{n+1}) = \cdots$. We claim that $J$ is nilpotent. If not, there exists a maximal element $r(a)$ in nonempty set $\{r(b) : bj^n \neq 0\}$. Assume that $J^{n+1} \neq 0$ and we get a contradiction. Since $l(J^n) = l(J^{2n})$, we have $J^{2n} \neq 0$. This implies that there exists an element $x \in J^n$ such that $axJ^n \neq 0$. Because of GP-injectivity of $R$, $l(J) \leq e \leq R$ and so $l(J^n) \leq e \leq R$ since $l(J) \leq l(J^n)$. Therefore there exists an element $y \in J^n$ such that $0 \neq yax \in l(J^n)$, and so $r(a) \leq r(ya)$. This is a contradiction of the maximality of $r(a)$. Hence $J$ is nilpotent by Hopkin's Theorem [1], so $R$ is a right Artinian ring.

In [3], Chen and Li proved that every right Noetherian, left CS and left $p$-injective ring is QF.

Theorem 2.2. If $R$ is a right Noetherian, left CS and left GP-injective ring, then $R$ is right Artinian.

Proof. Let $R$ be a right Noetherian, left CS and left GP-injective ring. By [3, Theorem 2.11], $R$ is a left finite dimensional ring. Hence $R$ is a right Artinian ring by Lemma 2.1.

Hence one may ask the following question.

Question: Let $R$ be a right Noetherian, left CS and left GP-injective ring. Is $R$ left Artinian?

If the answer is true, then $R$ is a QF-ring by [9, Theorem 3.4] because $\text{Soc (Re)}$ is simple for any local idempotent $e \in R$.

Recall that a ring $R$ said to be right Kasch ring if every simple right $R$-module embeds in $R$ and $R$ said to be a semiregular ring if $R/J$ is von Neumann regular and idempotents can be lifted modulo $J$. 

References for notions of rings and modules not defined in this work.
Theorem 2.3. [9, Theorem 3.31] Suppose that $R$ is a semilocal, left and right mininjective ring with ACC on right annihilators in which $S_r \leq R_R$. Then $R$ is a QF-ring.

Theorem 2.4. Let $R$ be a left GP-injective, left CS-ring with $S_i \leq R_R$ and right mininjective ring with ACC on right annihilators in which $S_r \leq R_R$. Then $R$ is QF-ring.

Proof. Let $e$ be any primitive idempotent of $R$. It is easy to see that $Re$ is uniform. This follows that $\text{Soc}(Re)$ is simple and so $R$ is left mininjective ring by [8]. Since $R$ is a left GP-injective ring, we have $J(R) = Z(R_R)$. By [9, Lemma 8.1], $R$ is a right Kasch ring and so $R$ is semiperfect by [9, Theorem 4.10]. By [9, Theorem 3.24] and [2, Theorem 2.3], $R$ is a left Kasch ring with $S_r = S_i$. Therefore $S_r \leq R_R$ by [2, Theorem 2.3]. Hence $R$ is a QF-ring by Theorem 2.3.

Remark: A ring $R$ said to be a CF-ring if every cyclic right $R$-module embeds in $R$. In [3], they shown that:
1) If $R$ is right CF, semiregular and $J \leq Z_r$, then $R$ is a right Artinian ring.
2) A right CF, semiregular and right p-injective ring is QF.

Lemma 2.5. Let $R$ be a left Kasch and right CF-ring. Then $R$ is a right Kasch, right Artinian (and so right Noetherian) and semilocal ring with $J = Z_R$.

Proof. See [4, Theorem 2.6].

Theorem 2.6. Assume that $R$ is a right CF-ring. Then $R$ is a QF-ring if the following are satisfied:
1) $R$ is semiregular and right GP-injective ring or;
2) $R$ is left Kasch ring or;
3) $R$ is semiregular and right mininjective ring with $S_r \leq R_R$.

Proof. (1) and (3) If $R$ is a right GP-injective and semiregular ring with $S_r \leq R_R$, then $J = Z_R$. By Remark, $R$ is right Artinian. Because of right mininjectivity of $R$, we have $R$ is a QF-ring by Theorem 2.3. (2) It follows from Lemma 2.5 and Theorem 2.2.

Theorem 2.7. Assume that $R$ is a right CF-ring and right mininjective ring. Then the following are equivalent:
1) $R$ is QF
2) $S_i$ is finitely generated as left $R$-module
3) $R$ is semisimple

Proof. (1) $\Rightarrow$ (2) Clear.
(2) $\Rightarrow$ (3) By assumption, $R$ is a left p-injective and right Kasch ring, and so $S_i = S_r$. It is enough to show that $J = J(S_i)$ and $R/J$ is semisimple. Let $x \in J(S_i)$. For maximal left ideal $I$ of $R$ and simple left ideal $A$ of $R$, we consider the isomorphism $f : R/I \rightarrow A$. Clearly, $f(R/I)x = f((R/I)x) = 0$, that is $Ax = 0$. This implies that $(R/I)x = 0$ and so $x \in I$. The other side is obvious. Hence $J = J(S_i)$. Now, since $S_i$ is finitely generated as left $R$-module, we write $S_i = Rx_1 \oplus Rx_2 \oplus \cdots \oplus Rx_n$, where each $Rx_i$ is a simple left ideal of $R$. Note that $J = r(S_i) = \bigcap_{i=1}^n r(x_i)$ and $g : R/J = R/r(S_i) = R/\bigcap_{i=1}^n r(x_i) \rightarrow R/\bigoplus_{i=1}^n r(x_i)$ is a monomorphism. Therefore $R/J$ is semisimple.
(3) $\Rightarrow$ (1) If $R$ is a semilocal, right mininjective and right CF-ring, then $R$ is quasi-Frobenius by [9, Theorem 8.11].

Lemma 2.8. Assume that $R$ is a right simple injective ring, $M \neq \oplus n R$ and $M_R \neq R_R$. If $M$ is a finitely generated right $R$-module then $M$ is semisimple.

Proof. Let $M = m_1R + m_2R + \cdots + m_nR$ be a finitely generated $R$-module and $F$ be a free $R$-module. Then we have the epimorphism $g : F \cong \oplus n R \rightarrow M \cong \oplus n R/\text{Ker}(f)$ defined by $f(x_i) = \sum_{i=1}^n m_i(x_i)$ where $f : F \rightarrow M$ is an epimorphism. Since $R$ is a right simple injective ring, there exists $h : \oplus n R \rightarrow \oplus n a_i R$. Then $\oplus n a_i R$ is semisimple and $\text{Ker}(h) \subseteq \text{Ker}(g)$. Since $\oplus n a_i R$ is semisimple and $\alpha : \oplus n a_i R \rightarrow M$ is an epimorphism, we can say that $M$ is semisimple.

Theorem 2.9. Assume that $R$ is a right (left) self-
injective ring, $M \neq \bigoplus_{n} R$ and $M_{R} \neq R_{R}$. Then,

1. Every finitely generated right (left) $R$-module is a right (left) Artinian and right (left) Noetherian module of finite length.

2. Every finitely generated right $R$-module is injective and projective.

**Proof.** (1) By Lemma 2.8.
(2) Let $R$ be a right self-injective ring and $M$ be a finitely generated $R$-module. By Lemma 2.8, $M$ is semisimple. This implies that every submodule of $M$ is a direct summand.

**Corollary 2.10.** Assume that $R$ is a right perfect and two sided self injective ring such that $\text{Soc}(eR) \neq 0$ for every local idempotent $e$ of $R$. Let $M \neq \bigoplus_{n} R$ and $M_{R} \neq R_{R}$. Then is a QF-ring.

**Proof.** By [9, Theorem 6.16], $R$ is right and left Kasch ring. By Theorems 6.19 and 6.20 in [9], the ring $R$ is finitely cogenerated. Now, by Theorem 2.9, $R$ is left Artinian. This implies that $R$ is a QF-ring.

### 3. REFERENCES


